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D-Lattices

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Difference lattices (D-lattices), which generalize Boolean algebras, orthomodular lattices as well as MV algebras, are studied.

1. INTRODUCTION

In 1992, a new algebraic structure of fuzzy sets, a *D-poset of fuzzy sets* (Kôpka, 1992), was introduced. A difference operation is a primary notion in this structure.

By the transfer of the properties of a difference of fuzzy sets on an arbitrary partially ordered set we obtained a new mathematical model, *a D-poset* (Kôpka and Chovanec, 1994), which generalizes orthoalgebras (Foulis *et al.*, 1992), orthomodular posets (Pták and Pulmannová, 1991), the set of all effects (Dvurečenskij and Pulmannová, 1994), as well as MV algebras (Chang, 1959) and Boolean algebras (Sikorski, 1964).

In this paper, we introduce D-lattices (i.e., D-posets which are simultaneously lattices) and we give a characterization of orthomodular lattices, Boolean algebras, and MV algebras in a difference poset setup.

2. D-POSETS AND D-LATTICES

Let $(\mathcal{P}, \leq)$ be a nonempty partially ordered set (poset). A partial binary operation $\setminus$ is called a *difference* on $\mathcal{P}$, if an element $b \setminus a$ is defined in $\mathcal{P}$ if and only if $a \leq b$, and the following conditions are satisfied:

(D1) If $a \leq b$, then $b \setminus a \leq b$.
(D2) If $a \leq b$, then $b \setminus (b \setminus a) = a$.
(D3) If $a \leq b \leq c$, then $c \setminus b \leq c \setminus a$ and $(c \setminus a) \setminus (c \setminus b) = b \setminus a$.

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A structure $(\mathcal{P}, \leq, \backslash)$ is called *a poset with a difference.* Moreover, if $\mathcal{P}$ is a lattice, we say that $(\mathcal{P}, \wedge, \vee, \backslash)$ is a *lattice with a difference.*

We shall write $\mathcal{P}$ instead of $(\mathcal{P}, \leq, \backslash)$ or $(\mathcal{P}, \wedge, \vee, \backslash)$.

It is easy to see that the set-theoretic difference of subsets of a nonempty set and the usual difference of nonnegative real numbers fulfill the conditions (D1)–(D3).

Example 2.1. Let $\mathcal{C} = \{x_0, x_1, \ldots, x_n, \ldots\}$ be an infinite countable chain such that $x_0 < x_1 < \cdots < x_n < \ldots, n \in N$. We put

$$x_k \backslash x_j := x_{k-j} \qquad \text{for} \quad k, j \in \{0, 1, 2, \ldots\}, \quad j \leq k$$

Then $\mathcal{C}$ is a lattice with a difference.

Proposition 2.2 (Kôpka and Chovanec, 1994). Let $\mathcal{P}$ be a poset with a difference and $a, b, c, d \in \mathcal{P}$. The following assertions are true:

(i) If $a \leq b \leq c$, then $b \backslash a \leq c \backslash a$ and $(c \backslash a) \backslash (b \backslash a) = c \backslash b$.
(ii) If $b \leq c$ and $a \leq c \backslash b$, then $b \leq c \backslash a$ and $(c \backslash b) \backslash a = (c \backslash a) \backslash b$.
(iii) If $a \leq b \leq c$, then $a \leq c \backslash (b \backslash a)$ and $(c \backslash (b \backslash a)) \backslash a = c \backslash b$.
(iv) If $a \leq c$ and $b \leq c$, then $c \backslash a = c \backslash b$ if and only if $a = b$.
(v) If $d \in \mathcal{P}$, $d \leq a \leq c$, $d \leq b \leq c$, then $c \backslash a = b \backslash d$ if and only if $c \backslash b = a \backslash d$.

Proposition 2.3. Let $\mathcal{P}$ be a poset with a difference and $a, b, c \in \mathcal{P}$, $a \leq c$, $b \leq c$. If $a \vee b \in \mathcal{P}$, then $(c \backslash a) \wedge (c \backslash b) \in \mathcal{P}$ and

$$c \backslash (a \vee b) = (c \backslash a) \wedge (c \backslash b)$$

Proof. From the inequalities $a \leq a \vee b \leq c$ and $b \leq a \vee b \leq c$ and (D3) we have $c \backslash (a \vee b) \leq c \backslash a$ and $c \backslash (a \vee b) \leq c \backslash b$. Let $w \in \mathcal{P}$, $w \leq c \backslash a$, $w \leq c \backslash b$. Then $a = c \backslash (c \backslash a) \leq c \backslash w$, $b = c \backslash (c \backslash b) \leq c \backslash w$; therefore, $a \vee b \leq c \backslash w \leq c$ and so $w = c \backslash (c \backslash w) \leq c \backslash (a \vee b)$, which implies that the element $c \backslash (a \vee b)$ is the greatest lower bound of the set $\{c \backslash a, c \backslash b\}$. ∎

Similarly we can prove the following proposition.

Proposition 2.4. Let $\mathcal{P}$ be a lattice with a difference and $a, b, c \in \mathcal{P}$, $a \leq c$, $b \leq c$. Then

$$c \backslash (a \wedge b) = (c \backslash a) \vee (c \backslash b)$$

Corollary 2.5. Let $\mathcal{P}$ be a lattice with a difference and $a, b \in \mathcal{P}$. Then

$$(a \vee b) \backslash (a \wedge b) = ((a \vee b) \backslash a) \vee ((a \vee b) \backslash b)$$

Proposition 2.6. Let $\mathcal{P}$ be a lattice with a difference and $a, b, c \in \mathcal{P}$, $c \leq a$, $c \leq b$. Then

$$(a \wedge b)\backslash c = (a\backslash c) \wedge (b\backslash c).$$

Proof. Calculate

$$\begin{aligned}(a \wedge b)\backslash c &= ((a \vee b)\backslash c)\backslash((a \vee b)\backslash(a \wedge b)) = \\ &= ((a \vee b)\backslash c)\backslash(((a \vee b)\backslash a) \vee ((a \vee b)\backslash b)) \\ &= (((a \vee b)\backslash c)\backslash((a \vee b)\backslash a)) \wedge (((a \vee b)\backslash c)\backslash((a \vee b)\backslash b)) \\ &= (a\backslash c) \wedge (b\backslash c). \quad \blacksquare\end{aligned}$$

If the greatest element $1_{\mathcal{P}}$ exists in a poset (a lattice) $\mathcal{P}$ with a difference, then $\mathcal{P}$ is called a D-poset (a D-lattice).

In Kôpka and Chovanec (1994) it was shown that every orthoalgebra, every orthomodular poset, and the set of all effects are D-posets. A Boolean algebra, an orthomodular lattice, and an MV algebra are D-lattices.

It is clear that the element $1_{\mathcal{P}} \backslash 1_{\mathcal{P}}$ is the least element in $\mathcal{P}$ and we denote it by $0_{\mathcal{P}}$.

Proposition 2.7 (Kôpka and Chovanec, 1994). Let $\mathcal{P}$ be a D-poset. Then:

(i) $a\backslash 0_{\mathcal{P}} = a$ for all $a \in P$.
(ii) $a\backslash a = 0_{\mathcal{P}}$ for all $a \in P$.
(iii) If $a, b \in P$, $a \leq b$, then $b\backslash a = 0_{\mathcal{P}}$ if and only if $b = a$.
(iv) If $a, b \in P$, $a \leq b$, then $b\backslash a = b$ if and only if $a = 0_{\mathcal{P}}$.

Proposition 2.8. Let $\mathcal{P}$ be a D-poset. If $a \vee b \in \mathcal{P}$, then $((a \vee b)\backslash a) \wedge ((a \vee b)\backslash b) \in \mathcal{P}$ and

$$((a \vee b)\backslash a) \wedge ((a \vee b)\backslash b) = 0_{\mathcal{P}}$$

Proof. It suffices to put $c = a \vee b$ in Corollary 2.5. ■

Proposition 2.9. Let $\mathcal{P}$ be a D-lattice, $a, b, c \in \mathcal{P}$, $c \leq a$, $c \leq b$. Then

$$(a \vee b)\backslash c = (a\backslash c) \vee (b\backslash c)$$

Proof. It is evident that $(a\backslash c) \vee (b\backslash c) \leq (a \vee b)\backslash c$. Calculate

$$\begin{aligned}&((a \vee b)\backslash c)\backslash((a\backslash c) \vee (b\backslash c)) \\ &\quad = (((a \vee b)\backslash c)\backslash(a\backslash c)) \wedge (((a \vee b)\backslash c)\backslash(b\backslash c)) \\ &\quad = ((a \vee b)\backslash a) \wedge ((a \vee b)\backslash b) = 0_{\mathcal{P}}\end{aligned}$$

and by (iii) in Proposition 2.7 we have $(a \vee b)\backslash c = (a\backslash c) \vee (b\backslash c)$. ■

Let $\mathcal{P}$ be a D-lattice. Then we can define a (total) binary operation $\ominus$ on $\mathcal{P}$ by the formula

$$b \ominus a := b\backslash(a \wedge b) \tag{2.1}$$

It is easy to prove that the binary operation $\ominus$ has the following properties:

(i) If $a \leq b$, then $b \ominus a = b \backslash a$.
(ii) $b \ominus a \leq b$ for any $a, b \in \mathcal{P}$.
(iii) $b \ominus (b \ominus a) = a \wedge b$.
(iv) If $b \leq a$, then $b \ominus a = 0_{\mathcal{P}}$.
(v) $a \wedge b = 0_{\mathcal{P}}$ if and only if $b \ominus a = b$.

Example 2.10. In Mundici (1986) an MV algebra is defined as follows: An *MV algebra* is an algebra $(\mathcal{A}, \oplus, \odot, *, 0, 1)$, where $\mathcal{A}$ is a nonempty set, 0 and 1 are constant elements of $\mathcal{A}$, $\oplus$ and $\odot$ are binary operations, and * is a unary operation, satisfying the following axioms:

(A1) $(a \oplus b) = (b \oplus a)$.
(A2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(A3) $a \oplus 0 = a$.
(A4) $a \oplus 1 = 1$.
(A5) $(a^*)^* = a$.
(A6) $0^* = 1$.
(A7) $a \oplus a^* = 1$.
(A8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.
(A9) $a \odot b = (a^* \oplus b^*)^*$.

The lattice operations $\vee$ and $\wedge$ are defined by the formulas

$$a \vee b = (a \odot b^*) \oplus b \qquad \text{and} \qquad a \wedge b = (a \oplus b^*) \odot b$$

We write $a \leq b$ iff $a \vee b = b$. The relation $\leq$ is a partial ordering over $\mathcal{A}$ and $0 \leq a \leq 1$, for every $a \in \mathcal{A}$. An MV algebra is a distributive lattice with respect to the operations $\vee$, $\wedge$. If we put

$$b \backslash a := (a \oplus b^*)^* \qquad \text{for} \quad a \leq b$$

then we obtain that an MV algebra is a D-poset, more exactly a distributive D-lattice. In Chovanec (1993) it is proved that

$$b \ominus a := b \backslash (a \wedge b) = b \odot a^* \qquad \text{for any} \quad a, b \in \mathcal{A}$$

Theorem 2.11. A D-lattice $\mathcal{P}$ is an MV algebra if and only if

$$(w \ominus u) \ominus v = (w \ominus v) \ominus u \qquad \text{for any} \quad u, v, w \in \mathcal{P}$$

Proof. Let $\mathcal{P}$ be an MV algebra. Then the formula (2.2) follows from the commutativity and the associativity of the operation $\odot$.

Now let $\mathcal{P}$ be a D-lattice with the property (2.2). We put

$$a^* := 1 \ominus a = 1 \backslash a \qquad \text{for any} \quad a \in \mathcal{P}$$

It is evident that * is an involution [axiom (A5)], an anti-isotonous operation on $\mathcal{P}$, and $0^* = 1$ [axiom (A6)]. Further, we define binary operations $\odot$ and $\oplus$ as follows:

$$a \odot b := a \ominus b^*$$

$$a \oplus b := (a^* \ominus b)^*$$

If we put $w = 1$, $u = a^*$, $v = b^*$ in (2.2), then

$$a \odot b = a \ominus b^* = (1 \ominus a^*) \ominus b^* = (1 \ominus b^*) \ominus a^* = b \ominus a^* = b \odot a$$

Therefore $\odot$ is a commutative operation and dually $\oplus$ is also commutative [axiom (A1)]. Similarly $\odot$ and $\oplus$ are associative operations [axiom (A2)]. Calculate

$$a \oplus 0 = (a^* \ominus 0)^* = (a^* \backslash 0)^* = (a^*)^* = a$$

$$a \oplus 1 = (a^* \ominus 1)^* = 0^* = 1$$

$$a \oplus a^* = (a^* \ominus a^*)^* = (a^* \backslash a^*)^* = 0^* = 1$$

which proves the validity of axioms (A3), (A4), and (A7). Finally we prove axiom (A8). It is clear that $b \ominus (b \ominus a) = a \wedge b = a \ominus (a \ominus b)$. Then

$$\begin{aligned} a \oplus (a \oplus b^*)^* &= (a^* \odot (a \oplus b^*))^* = (a^* \ominus (a^* \ominus b^*))^* \\ &= (b^* \ominus (b^* \ominus a^*))^* = (b^* \odot (b \oplus a^*))^* \\ &= b \oplus (b \oplus a^*)^* \quad \blacksquare \end{aligned}$$

Corollary 2.12. An orthomodular lattice is a Boolean algebra if and only if the formula (2.5) holds for every trinity of elements from an orthomodular lattice.

Example 2.13. (a) Every finite chain is an MV algebra. Indeed, if $\mathcal{C} = \{x_0, x_1, \ldots, x_n\}$, $n \in N$, is a chain such that $x_0 < x_1 < \cdots < x_n$, then a difference \ on $\mathcal{C}$ is uniquely defined (see also Riečanová and Bršel, 1994) by

$$x_k \backslash x_j := x_{k-j} \qquad \text{for} \quad k, j \in \{0, 1, 2, \ldots, n\}, \quad j \leq k$$

The chain $\mathcal{C}$ is a distributive D-lattice with the least element x_0 and the greatest element x_n. It is easy to verify that the binary operation $\ominus$ on $\mathcal{C}$ defined by the formula (2.1) has the property (2.2).

(b) Let $(\mathcal{G}, +)$ be an ordered Abelian group and $e \in \mathcal{G}$, $e > 0$. Then the set (the interval) $\mathcal{G}_{[0,e]} = \{a \in \mathcal{G}: 0 \leq a \leq e\}$ is an MV algebra.

(c) Let $\mathcal{D} = \{0, 1, a, b, c, d, e\}$ be a system which arose from two chains $\{0, b, a, 1\}$ and $\{0, e, d, c, 1\}$. If we put $1\backslash a = b$, $1\backslash b = a$, $a\backslash b = b$, $1\backslash e = c$, $1\backslash c = e$, $1\backslash d = d$, $c\backslash d = e$, $c\backslash e = d$, $d\backslash e = e$, $x\backslash 0 = x$, and $x\backslash x = 0$ for any $x \in \mathcal{C}$, then $\mathcal{C}$ is a D-lattice, which is not an MV algebra. Indeed, $(1 \ominus c) \ominus a = e\backslash(e \wedge a) = e$ and $(1 \ominus a) \ominus c = b\backslash(b \wedge c) = b$.

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